Discrete Mathematics 119 (1993) 49–66
North-Holland

The even and odd cut polytopes

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Received 6 June 1990
Revised 10 March 1992

Abstract

Deza, M. and M. Laurent, The even and odd cut polytopes, Discrete Mathematics 119 (1993) 49–66.

The cut polytope P_n is the convex hull of the incidence vectors of all cuts of the complete graph K_n on n nodes. An even cut is a cut of even cardinality. For n odd, all cuts are even. For n even, we consider the even cut polytope EvP_n, defined as the convex hull of the incidence vectors of all even cuts of K_n. The vertex sets of both polytopes P_n and EvP_n come from the cycle sets of some binary matroids; a more general example is the even T-cut polytope. We study the facial structure of EvP_n and of the closely related (via switching) odd cut polytope. We give the description of their symmetry groups, show that all facets of P_n can be zero-lifted to EvP_m for $m \geqslant n+5$ and also exhibit two classes of facet inducing valid inequalities of EvP_n. One is an even analogue of the hypermetric inequalities for P_n and the other consists of inequalities violated by exactly one odd cut.

1. Introduction

We denote by K_n the complete graph on the n nodes $[1,n]=\{1,\dots,n\}$. Given a subset S of $[1,n]$, the *cut* defined by S is the set $\delta(S)$ of the edges (i,j) of K_n having exactly one endnode in S; the sets S and $[1,n]-S$ are called the *shores* of the cut $\delta(S)$. The incidence vector of the cut $\delta(S)$ is the vector $X^{\delta(S)}$ of $\mathbf{R}^{n(n-1)/2}$ defined by $X_{ij}^{\delta(S)}=1$ if $|S\cap\{i,j\}|=1$ and $X_{ij}^{\delta(S)}=0$ otherwise, for $1\leqslant i<j\leqslant n$. We are interested here in some particular cuts specified by conditions on the cardinality of their shores. For n even, a cut $\delta(S)$ is called an *even cut* (resp. *odd cut*) if its shores S and $[1,n]-S$ are of even (resp. odd) cardinality or, equivalently, the cut $\delta(S)$ is of even cardinality. We consider only the case in which n is even, because, for n odd, any cut has shores of distinct parities and so is of even cardinality. The cut polytope P_n is the convex hull of the incidence vectors of all cuts of K_n and, for n even, the even cut polytope EvP_n (resp. odd cut polytope OdP_n) is the convex hull of the incidence vectors of all even cuts (resp. odd cuts) of K_n. Similarly, the cut cone C_n (resp. the even cut cone EvC_n, the odd

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cut cone OdC_n) is the cone generated by the incidence vectors of all cuts (resp. even cuts, odd cuts).

The cut polytope and the cut cone are important tools in a number of combinatorial settings, in particular, for the polyhedral approach to the maximum cut problem and for the characterization of finite metric spaces that are isometrically embeddable into l_1. They were studied in many papers, e.g. [5, 8–10, 15, 16]. An important feature is that cuts are the cycles of a binary matroid; so the cut polytope is a special case of cycle polytope of binary matroid introduced in [4]. In fact, even cuts are also the cycles of a binary matroid. So the even cut polytope is another example of cycle polytope of binary matroid for which, however, we could still take advantage of the graph structure. This feature was one of the motivations of this paper.

Another motivation comes from the application to the maximum even cut problem. The maximum cut problem is NP-hard [18]. In fact, it can be reduced to the maximum even cut problem which is, therefore, also NP-hard (see Section 3). This fact can be put in parallel with the fact that the facial structure of the even cut polytope is more complicated than that of the cut polytope. Indeed, we show that every facet of the cut cone C_n can be extended to a facet of the even cut cone EvC_m, for even $m \geqslant n+5$ (see Theorem 3.1).

On the other hand, the full characterization of both the cut lattice and the even cut lattice is known. The cut lattice (or even cut lattice) consists of all linear combinations of cuts (or even cuts) with integer coefficients. The description of the cut lattice is given in [1] and that of the even cut lattice can be found in [17].

The paper is organized as follows. In Section 2, we present the connections between the even cut polytope, the odd cut polytope and their respective cones, using the general switching operation for the cycle polytopes of binary matroids from [4]. This connection is also stated in the more general framework of T-cuts. The main result of the section is that the only symmetries of the even cut polytope come from permutations and switchings (by even cuts). We also point out that, besides the facets coming from the triangle inequalities, no other class of facets for the even cut polytope comes from some known class of facets for the general cycle polytope. The main result of Section 3 is that every facet of the cut cone C_n yields (by zero-lifting) a facet of the even cut cone EvC_m for m even, $m \geqslant n+5$. Therefore, the description of the even cut polytope is an even more complicated task than that of the cut polytope. Sections 4 and 5 describe two classes of valid inequalities and facets of the even cut cone which do not come from the cut cone. The first class is an even analogue of the hypermetric inequalities for the cut one, while the second class consists of inequalities violated by exactly one odd cut. We also mention generalizations of the first class for even T-cuts, σ-ary cuts and even multicuts. Section 6 contains the proofs of the main results.

We now give some preliminaries and notation needed for the paper. For a vector $v \in \mathbf{R}^{n(n-1)/2}$ and a subset E of edges of K_n, we set $v(E) = v.X^E = \sum_{ij \in E} v_{ij}$. For an n-vector $b = (b_1, \ldots, b_n)$ and a subset S of $[1, n]$, we set $b(S) = \sum_{i \in S} b_i$. Given a subset

X of $\mathbf{R}^{n(n-1)/2}$, its dimension $\dim(X)$ is the maximum number of affinely independent vectors in X minus one, and, if X contains the origin, $\dim(X)$ is also the maximum number of linearly independent vectors in X. If $\dim(X)=n(n-1)/2$, then X is full dimensional. Given a vector $v\in\mathbf{R}^{n(n-1)/2}$, a scalar $v_0\in R$, the inequality $v.x\leqslant v_0$ is valid for the set X if it is satisfied by all vectors x of X, i.e. by the incidence vectors of all cuts (resp. even cuts, odd cuts) if $X=P_n$ or C_n (resp. EvP_n or EvC_n, OdP_n or OdC_n). The cuts (resp. even cuts, odd cuts) whose incidence vectors satisfy equality $v.x=v_0$ are called the *roots* of the inequality $v.x\leqslant v_0$. If X is a cone or a polytope and $v.x\leqslant v_0$ is valid for X, the set $F=\{x\in X;\ v.x=v_0\}$ is the face of X induced by the inequality $v.x\leqslant v_0$ and F is a facet of X if $\dim(F)=\dim(X)-1$. If X is a cone pointed at the origin (e.g. C_n, EvC_n), then its facets are supported by homogeneous inequalities, i.e. of the form $v\cdot x\leqslant 0$. Note that P_n, C_n for $n\geqslant 3$, $\mathrm{EvP}_n, \mathrm{EvC}_n, \mathrm{OdP}_n, \mathrm{OdC}_n$ for n even, $n\geqslant 6$, are full dimensional. Indeed, it can be checked that both families $\{X^{\delta(\{i,j\})}, 1\leqslant i<j\leqslant n\}$ and $\{X^{\delta(\{i\})}, 2\leqslant i\leqslant n,\ X^{\delta(\{1,i,j\})}, 2\leqslant i<j\leqslant n\}$ are linearly independent.

2. Relation to binary matroids

Let M be an $r\times s$ matrix with zero–one coefficients. The matrix M defines a binary matroid, also denoted by M, on the index set $[1,s]$ of the columns of M as follows. A subset S of $[1,s]$ is independent in M if the corresponding set of columns of M is linearly independent over the field $GF(2)$, and S is a circuit of M if S is not independent but every proper subset of S is independent. A subset C of $[1,s]$ is a *cycle* of M if $C=\emptyset$ or C is a disjoint union of circuits of M, i.e. C is dependent in M. So, the cycles of M correspond precisely to the zero–one solutions $x\in\{0,1\}^s$ of the system of congruences $Mx\equiv 0 \pmod 2$. The *cycle polytope* $P(M)$ [4] of thé binary matroid M is the convex hull of the incidence vectors of the cycles of M, i.e. $P(M)=\mathrm{Conv}(\{x\in\{0,1\}^s\colon Mx\equiv 0 \pmod 2\})$. Given any vector $b\in\{0,1\}^s$, the polytope $P(M,b)=\mathrm{Conv}(\{x\in\{0,1\}^s\colon Mx=b \pmod 2\})$ was also considered in [4]; so $P(M,0)=P(M)$.

The dual matroid M^* of the matroid M on $[1,s]$ is the matroid whose maximal independent sets are the complements in $[1,s]$ of the maximal independent sets of M; then, the circuits of M^* are called the cocircuits of M. For subsets Z,Z' of $[1,s]$, $M\backslash Z$ denotes the matroid obtained by deleting Z (whose circuits are the circuits of M disjoint from Z) and M/Z' denotes the matroid obtained by contracting Z' (whose cocircuits are the cocircuits of M disjoint from Z'); then, $M\backslash Z/Z'$ is a minor of M. The dual Fano matroid F_7^* is the binary matroid on 7 points defined by the matrix whose rows are the four vectors $(0,1,1,1,0,0,0)$, $(1,0,1,0,1,0,0)$, $(1,1,0,0,0,1,0)$ and $(1,1,1,0,0,0,1)$.

Barahona and Grötschel [4] introduced an operation, called *switching* operation, and defined as follows. Given a vector $v\in\mathbf{R}^s$ and a subset D of $[1,s]$, define the vector $v^D\in\mathbf{R}^s$ by $v_e^D=-v_e$ if $e\in D$ and $v_e^D=v_e$ if $e\notin D$. Given an inequality $v.x\leqslant v_0$, one says that the inequality $v^D.x\leqslant v_0-v(D)$ is obtained by *switching the inequality* $v.x\leqslant v_0$ *by the set* D.

We recall below some applications of the switching operation, described in [4].

The polytope $P(M,b)$ can be obtained from the polytope $P(M)$ using this switching operation. Namely, let $v\in\mathbf{R}^s$, $v_0\in\mathbf{R}$ and let D be a subset of $[1,s]$ such that X^D belongs to the polytope $P(M,b)$, i.e. $MX^D\equiv b \pmod 2$. The inequality $v.x\leqslant v_0$ is valid (facet inducing) for the polytope $P(M)$ if and only if the inequality $v^D.x\leqslant v_0-v(D)$ is valid (facet inducing) for the polytope $P(M,b)$.

Another application of the switching operation is that all the facets of the polytope $P(M)$ can be obtained from the homogeneous facets of $P(M)$, i.e. from the facets of the cone generated by the incidence vectors of the cycles of the matroid M. Indeed, for any $v\in\mathbf{R}^s$, $v_0\in\mathbf{R}$ and for any cycle C of M such that $v(C)=v_0$, the inequality $v.x\leqslant v_0$ is valid (facet inducing) for the polytope $P(M)$ if and only if the inequality $v^C.x\leqslant 0$ is valid (facet inducing) for $P(M)$.

If M is a cographic matroid, e.g. the rows of the matrix M are the incidence vectors of the cycles (in graph terminology) of a graph G, then $P(M)$ is precisely the cut polytope of the graph G. The even cut polytope of a graph G is the cycle polytope $P(M_e(G))$ for the binary matroid $M_e(G)$, called *even cut matroid*, defined by the matrix whose rows are the incidence vectors of the cycles of G together with one more last row equal to the vector $(1,\dots,1)$ of all ones. By even cut, we mean then a cut of even cardinality, and so this matroid belongs to the subclass of binary matroids whose cycles are all of even cardinality. On the other hand, the odd cut polytope of G is the polytope $P(M_e(G),b)$, where the vector b is equal to $(0,\dots,0,1)$.

We have the following facts from the above discussion:

(2.1) The facets of the odd cut polytope are exactly the switchings, by odd cuts, of the facets of the even cut polytope and vice versa.

(2.2) Any switching of a facet of the even cut polytope (or the odd cut polytope) by an even cut is a facet of the even cut polytope (or the odd cut polytope).

(2.3) Any facet of the even cut polytope is obtained by switching by an even cut a facet of the even cut cone.

Take again the cycle polytope $P(M)$ for some binary matroid M on $[1,s]$. Given a subset C of $[1,s]$, let r_C denote the reflection of the space $\mathbf{R}^s$ defined by $r_C(x)=y$ with $y_e=1-x_e$ if $e\in C$ and $y_e=x_e$ otherwise, for $e\in[1,s]$. If C,D are two cycles of the binary matroid M, then $r_C(x^D)=X^C+X^D \pmod 2 = X^{C\Delta D}$ holds. Hence, for any cycle C of M, the reflection r_C preserves the cycle polytope $P(M)$ and thus is a symmetry of $P(M)$. In particular, for any cut $\delta(S)$, the reflection $r_{\delta(S)}$ is a symmetry of the cut polytope while, for any even cut $\delta(S)$, $r_{\delta(S)}$ is a symmetry of both the even and odd cut polytopes. For $v\in R^s$ and for C,D cycles of M, the following relation holds:

$$v^C(C\,\Delta\, D)=v(D)-v(C) \tag{1}$$

Hence, if the inequality $v.x\leqslant v_0$ defines a facet F of the polytope $P(M)$, then the inequality $v^C.x\leqslant v_0-v(C)$ defines the facet $r_C(F)$ of $P(M)$. Therefore, the reflection r_C corresponds, in fact, to the switching by cycle C operation. Hence, the switchings by cycles form a subgroup of the symmetry group of any cycle polytope $P(M)$. We now turn to the full description of the symmetry groups of the even and odd cut polytopes in the case of the complete graph.

Let K_n be a complete graph with n even. We denote by $\mathrm{Is}(\mathrm{EvP}_n)$ (resp. $\mathrm{Is}(\mathrm{OdP}_n)$) the group of symmetries of the polytope EvP_n (resp. OdP_n), i.e. the group of the isometries of the space $\mathbf{R}^{n(n-1)/2}$ preserving the polytope EvP_n (resp. OdP_n). As mentioned above, the set $R_n=\{r_{\delta(S)}: \delta(S)$ is an even cut of $K_n\}$ is a subgroup of $\mathrm{Is}(\mathrm{EvP}_n)$ and of $\mathrm{Is}(\mathrm{OdP}_n)$. Every permutation σ of $[1,n]$ induces naturally an isometry of $\mathbf{R}^{n(n-1)/2}$ defined by $\sigma(x)=(x_{\sigma(i)\sigma(j)})_{1\leqslant i<j\leqslant n}$ for all $x\in\mathbf{R}^{n(n-1)/2}$. Since $\sigma(\delta(S))=\delta(\sigma^{-1}(S))$ for any subset S of $[1,n]$, every permutation σ of $\mathrm{Sym}(n)$ induces a symmetry of all polytopes P_n, EvP_n, OdP_n. So, we have another subgroup $\mathrm{Sym}(n)$ of $\mathrm{Is}(P_n)$, $\mathrm{Is}(\mathrm{EvP}_n)$ $\mathrm{Is}(\mathrm{OdP}_n)$. It was proved [12] that permutations and switchings (by all cuts) are the only symmetries of the cut polytope P_n, for $n\neq 4$. We prove a similar result for the even and odd cut polytopes.

Let $G_n=\mathrm{Sym}(n)\cdot R_n$ denote the group generated by all permutations $\sigma\in\mathrm{Sym}(n)$ and all reflections $r_{\delta(S)}$ for $\delta(S)$ even cut of K_n; G_n is a semidirect product whose commutation rule follows from:

$$r_{\delta(S)}\sigma=\sigma r_{\delta(\sigma(S))}\quad \text{for } \sigma\in\mathrm{Sym}(n) \text{ and } \delta(S) \text{ cut of } K_n. \tag{2}$$

Theorem 2.4. *For n even, both the even cut polytope* EvP_n *and the odd cut polytope* OdP_n *have the same symmetry group which is equal to*:

(i) $G_n=\mathrm{Sym}(n)\cdot 2^{n-2}$ *for* $n\geqslant 8$;

(ii) $G_n=\mathrm{Sym}(2^{n-2})$ *for* $n=2,4$;

(iii) $\mathrm{Sym}(2^{n-2})$ (*containing strictly* G_n) *for* $n=6$.

Proof. Let us first rule out the two easy cases $n=2,6$. If $n=2$, then both EvP_2, OdP_2 are reduced to a single point, and thus their symmetry group consists only of the identity map. If $n=6$, then both EvP_6, OdP_6 are full-dimensional regular simplices in $\mathbf{R}^{15}$ (with side length $\sqrt{}8$), and thus their symmetry group is $\mathrm{Sym}(16)$.

We have the following statement:

$$|G_n|=n!\,2^{n-2}\quad \text{if } n\neq 4 \text{ and } |G_4|=4! \tag{3}$$

For this, note first that $|R_n|=2^{n-2}$ and $R_n\cap\mathrm{Sym}(n)=\{\mathrm{id}\}$. Let σ be a permutation which acts as the identity on EvP_n, i.e. $\sigma(\delta(S))=\delta(\sigma^{-1}(S))=\delta(S)$, and so $\sigma^{-1}(S)=S$ or $[1,n]-S$ for all even cuts $\delta(S)$. Note that $\sigma^{-1}(S)=[1,n]-S$ can occur only if $|S|=n/2$. If $n\neq 4$, then $\sigma(\delta(\{i,j\}))=\delta(\{i,j\})$ for $1\leqslant i<j\leqslant n$ and thus $\sigma=\mathrm{id}$. Hence, for $n\neq 4$, $|G_n|=n!\,2^{n-2}$. In the case $n=4$, it may happen that $\sigma(\{1,2\})=\{3,4\}$; one checks easily that the only permutations that act as the identity on $\mathrm{Is}(\mathrm{EvP}_4)$ are (12)(34), (13)(24) and id, yielding that $|G_4|=4!/4\ 2^2=4!$.

We have a subgroup G_n of $\mathrm{Is}(\mathrm{EvP}_n)$. We now exhibit a (known) group containing $\mathrm{Is}(\mathrm{EvP}_n)$ as subgroup. For this, consider the graph H_n whose vertices are the even cuts of K_n and two even cuts $\delta(S)$, $\delta(T)$ are adjacent if $|S \Delta T| = 2$ or $n-2$, or equivalently, $\|\delta(S)-\delta(T)\|^2 = 2(n-2)$, where $\|x\|^2 = \sum_i (x_i)^2$. By definition, every symmetry of EvP_n induces an automorphism of the graph H_n, so $\mathrm{Is}(\mathrm{EvP}_n)$ is a subgroup of $\mathrm{Aut}(H_n)$. So we have the inclusions $G_n \subseteq \mathrm{Is}(\mathrm{EvP}_n) \subseteq \mathrm{Aut}(H_n)$. The graph H_n is known as the folded half-cube and its automorphism group is known (see [6, p. 265]); $\mathrm{Aut}(H_n) = 2^{n-2}\mathrm{Sym}(n)$ if $n \geq 8$ and $\mathrm{Aut}(H_n) = \mathrm{Sym}(2^{n-2})$ for $n = 2, 4, 6$. Therefore, if $n \geq 8$ or $n = 4$, then both groups G_n, $\mathrm{Aut}(H_n)$ have the same size, implying that $\mathrm{Is}(\mathrm{EvP}_n) = G_n$.

Finally, using (2.1), if $\delta(S)$ is a given odd cut, then $\mathrm{OdP}_n = r_{\delta(S)}(\mathrm{EvP}_n)$ from which one deduces that $\mathrm{Is}(\mathrm{OdP}_n) = r_{\delta(S)}\, \mathrm{Is}(\mathrm{EvP}_n) r_{\delta(S)}$ and, using the commutation rule (2), $\mathrm{Is}(\mathrm{OdP}_n) = \mathrm{Is}(\mathrm{EvP}_n)$ □

Actually, even and odd cuts are a very special case of the more general notion of even and odd T-cuts. We recall the definitions for the case of any graph $G(V, E)$ with nodeset V and edgeset E. Let T be a subset of V of even cardinality; a cut $\delta(S)$ is called an *even T-cut* (resp. *odd T-cut*) if $|S \cap T|$ is an even (resp. odd) number. A subset H of E is called a *T-join* if the nodes in T are precisely the nodes of V having odd degree in the subgraph (V, H). T-joins and T-cuts turn out to be a suitable generalization of several interesting special cases; for details, see, e.g. [19, 20]. Barahona and Conforti [2] proved that the family of even T-cuts of the graph G is the family of cycles of a binary matroid $M(G, T)$, called *T-join matroid*, whose cocycles are the T-joins of G and the Eulerian subgraphs of G. This can be seen as follows. For any cut $\delta(S)$, the sets $H \cap \delta(S)$ and $S \cap T$ have the same parity (this follows from the relation: $\sum_{i \in S} \deg_H(i) = \sum_{i \in S \cap T} \deg_H(i) + \sum_{i \in S - T} \deg_H(i) = |H \cap \delta(S)| + 2|H \cap E(S)|$, where $E(S)$ denotes the set of edges contained in S). Therefore, a cut $\delta(S)$ is an even T-cut (resp. odd T-cut) if and only if its intersection with any or all T-join is of even (resp. odd) cardinality. Hence, a matrix defining the T-join matroid $M(G, T)$ is the matrix whose rows are the incidence vectors of the cycles of G and one more last row equal to the incidence vector of a T-join of G. So, for our case $G = K_n$ with n even, even T-cuts with $T = V = (1, n)$ coincide with even cuts (and $H = K_n$ is a T-join), and thus the V-join matroid $M(K_n, V)$ coincides with the even cut matroid $M_e(K_n)$. Just as the odd cut polytope can be obtained from the even cut polytope by switching by an odd cut, the odd T-cut polytope can be obtained from the even T-cut polytope by switching by an odd T-cut. We shall see in Section 4 how some classes of valid inequalities for the even cut polytope can be generalized to the even T-cut polytope.

We conclude this section by pointing out that the only class of (nontrivial) facets for the even cut polytope EvP_n that could be deduced from some general class of facets known for the cycle polytope of any binary matroid are the facets induced by the triangle inequalities. Actually, the only facet inducing inequalities known for the general cycle polytope are the inequalities (4) below coming from the cocircuits of the binary matroid M. If C is cocircuit of M and e is an element not belonging to C, one says that e is a *chord* of C if there exist cocircuits D, E of M such that $D \cap E = \{e\}$ and

$D \Delta E = C$. Let C be a cocircuit of M of size at least 3 and let e be an element of C. If C has no chord and if the matroid M has no F_7^* minor, then the inequality:

$$x_e - \sum_{f \in C-\{e\}} x_f \leqslant 0 \tag{4}$$

defines a facet of the cycle polytope $P(M)$ [4]. We now examine what are the implications of this fact for the even cut polytope EvP_n. A first observation is that the dual Fano matroid F_7^* is indeed a minor of the even cut matroid $M_e(K_n) = M(K_n, V)$ ($V = [1, n]$). To see this, let H denote the graph on nodes $(1, 6)$ with the pairs $(1, 2)$, $(3, 4)$, $(3, 5)$, $(3, 6)$, $(4, 5)$, $(4, 6)$ and $(5, 6)$ as edges; let Z denote the set of edges (i, j) with $7 \leqslant i < j \leqslant n$ and let Z' denote the set of edges (i, j) with $1 \leqslant i \leqslant 2$ and $3 \leqslant j \leqslant n$, or $3 \leqslant i \leqslant 6$ and $7 \leqslant j \leqslant n$. Both matroids $M(H, [1, 6])$ and $M_e(H)$ coincide with the dual Fano matroid F_7^* (see also [2]). Also, one easily checks that the minor $M(K_n, V)\backslash Z/Z'$ coincides with $M(H, [1, 6])$, i.e. with F_7^*. So one of the conditions for claiming that (4) is facet inducing is not fulfilled. Note then that the cocircuits of $M_e(K_n) = M(K_n, V)$ are the circuits of K_n and the minimal V-joins of K_n. However, the only chordless circuits of K_n are the triangles and then the corresponding inequality (4) is the triangle inequality that is indeed facet inducing for the even cut polytope EvP_n if $n \geqslant 8$ (see Theorem 3.1 and the sentence following it). On the other hand, one easily checks that the minimal V-joins of K_n, i.e. the minimal edgesets H for which $\deg_H(i)$ is odd for all i in V are precisely the forests of K_n in which every node has odd degree. Such a forest H is chordless if and only if H is a perfect matching. Indeed, assume that, e.g., node 1 is adjacent to nodes 2, 3 in H; let T denote the triangle $(1, 2, 3)$ and set $D = H - \{(1, 2), (1, 3)\} + \{(2, 3)\}$, then D is a V-join, $T \cap D = \{(2, 3)\}$, $T \Delta D = H$, i.e. edge $(2, 3)$ is a chord of H. Therefore, the only cocircuits (besides the triangles) of $M(K_n, V)$ that could yield facets of EvP_n are the perfect matchings. However, the face of EvP_n defined by the corresponding inequality $x_{12} - (x_{34} + x_{56} + \cdots + x_{n-1n}) \leqslant 0$ is contained in the hyperplane of equation $x_{13} + x_{23} - x_{14} - x_{24} = 0$ and hence is not a facet of EvP_n.

3. Relation to the cut polytope

In the remainder, n always denotes an even integer. Let us first observe that the family of even cuts of K_n is in bijection with the family of all cuts of K_{n-1}. Namely, for a subset S of $[1, n-1]$, let x denote the incidence vector of the cut of K_{n-1} with shores S and $[1, n-1] - S$ and suppose, for instance, that S has even size (else replace S by $[1, n-1] - S$). Then, the cut of K_n with shores S and $[1, n] - S$ is an even cut whose incidence vector y is given by $y_{ij} = x_{ij}$ if $1 \leqslant i < j \leqslant n-1$ and $y_{in} = \sum_{1 \leqslant j \leqslant n-1, j \neq i} x_{ij} \pmod 2$ (this correspondence is similar to 'parity check' in error correcting codes). Note, however, that the above bijection is not linear and so does not extend to a bijection between the corresponding polytopes or cones. But, using this bijection, we can reduce the maximum cut problem to the maximum even cut problem. Indeed, let w be a weight function defined on the edges of K_{n-1}. Define w' on the edges of K_n by $w'_{ij}=w_{ij}$ for $1\leqslant i<j\leqslant n-1$ and $w'_{in}=0$ for $1\leqslant i\leqslant n-1$. Let $\delta(S)$ be an even cut in K_n whose weight $w'(\delta(S))$ is maximum and suppose, for instance, that $n\notin S$. As before, denote by y the incidence vector of the cut $\delta(S)$ of K_n and by x the incidence vector of the cut of K_{n-1} with shores S and $[1,n-1]-S$; then, $w.x=w'.y$ holds. We deduce that the cut of K_{n-1} with shores S and $(1,n-1)-S$ is an optimum solution for the maximum cut problem on K_{n-1} with weight function w. Therefore, the maximum even cut problem is NP-hard. Using switching, the maximum odd cut problem is also NP-hard.

Note that, for $n\geqslant 6$, the even cut polytope and cone are full dimensional; indeed, the $n(n-1)/2$ cuts $\delta(\{i,j\})$ for $1\leqslant i<j\leqslant n$ are linearly independent.

For $n=6$, EvC_6 is a simplicial cone and EvP_6 is a simplex in $\mathbf{R}^{15}$ whose linear descriptions are as follows [11]. Set

$$v_{ij}\cdot x=x_{ij}+\sum_{1\leqslant h<k\leqslant 6,\ h,k\neq i,j} x_{hk}-\sum_{1\leqslant k\leqslant 6,\ k\neq i,j,\ h\in\{i,j\}} x_{hk}$$

for $1\leqslant i<j\leqslant 6$. Then, $\mathrm{EvC}_6=\{x\in\mathbf{R}^{15}_+: v_{ij}\cdot x\leqslant 0 \text{ for } 1\leqslant i<j\leqslant 6\}$ and $\mathrm{EvP}_6=\{x\in\mathbf{R}^{15}_+: \sum_{1\leqslant i<j\leqslant 6}x_{ij}=8\}\cap\mathrm{EvC}_6$. Moreover, the lattice points of EvC_6, i.e. the points $x\in\mathbf{R}^{15}$ which can be written as linear combination with nonnegative integer coefficients of even cuts, are characterized by $v_{ij}.x\leqslant 0$ and $v_{ij}.x\equiv 0 \pmod 8$ for all $1\leqslant i<j\leqslant 6$ [11].

For $n=4$, EvC_4, EvP_4 have dimension 3 and are given in [11]

$$\mathrm{EvC}_4=\{x\in R^6: x_{12}+x_{13}-x_{23}\geqslant 0,\ x_{12}+x_{14}-x_{24}\geqslant 0,\ x_{13}+x_{14}-x_{34}\geqslant 0$$

$$x_{13}+x_{14}+x_{34}=x_{23}+x_{24}+x_{34}=x_{12}+x_{14}+x_{24}=x_{23}+x_{34}+x_{24}\}$$

$$\mathrm{EvP}_4=\mathrm{EvC}_4\cap\{x\in\mathbf{R}^6: x_{23}+x_{24}+x_{34}=2\}.$$

Some features of the cut polytope can be extended for the even cut polytope. This is the case, for instance, for the collapsing operation, considered in [8]. Given integers $n\leqslant m$, a partition π of $[1,m]$ into subsets $I_1,\ldots,I_n$ and a vector $v\in\mathbf{R}^{m(m-1)/2}$, the vector v_π of $\mathbf{R}^{n(n-1)/2}$, obtained by *π-collapsing* v, is defined by

$$(v_\pi)_{ij}=\sum_{h\in I_i,\ k\in I_j} v_{hk}$$

for $1\leqslant i<j\leqslant n$. For any subset S of $[1,n]$, we define the subset $S^\pi=\bigcup_{i\in S}I_i$ of $[1,m]$; then, $v_\pi(\delta(S))=v(\delta(S^\pi))$ holds (of course, $\delta(S)$ denotes the cut in K_n while $\delta(S^\pi)$ denotes the cut in K_m). Hence, if the inequality $v.x\leqslant v_0$ is valid for the cut polytope P_m, then its π-collapse $v_\pi.x\leqslant v_0$ is valid for the cut polytope P_n. Clearly, if all classes $I_1,\ldots,I_n$ of the partition π have the same parity, then $\delta(S^\pi)$ is an even cut whenever $\delta(S)$ is an even cut. Therefore, for n,m even, if the inequality $v.x\leqslant v_0$ is valid for EvP_m, then its π-collapse $v_\pi.x\leqslant v_0$ is valid for EvP_n.

On the other hand, the facets of the cut polytope have the following property: their support graph must be 2-connected [7]. This property is lost for the even cut polytope. For instance, the inequality $x_{12}-\sum_{3\leqslant i<j\leqslant 8}x_{ij}+2(x_{45}+x_{56}+x_{67}+x_{78}+x_{48})\leqslant 0$ defines a facet of EvC_8 whose support graph is disconnected.

Any inequality valid for the cut polytope is trivially valid for the even cut cone. The main result of this section is Theorem 3.1 showing that any facet of the cut cone C_n yields (by zero-lifting) a facet of the even cut cone EvC_m for $m \geqslant n+5$. In some sense, it means that the description of the even cut cone is even more complicated than that of the cut cone. The proof is given in Section 6.1.

Theorem 3.1. *Let $m \geqslant n+5$ be integers, with m even, $v \in \mathbf{R}^{n(n-1)/2}$ and $v' \in \mathbf{R}^{m(m-1)/2}$ defined by $v'_{ij} = v_{ij}$ for $1 \leqslant i < j \leqslant n$ and $v'_{ij} = 0$ for $1 \leqslant i \leqslant n < j \leqslant m$ or $n < i < j \leqslant m$. If the inequality $v.x \leqslant 0$ defines a facet of the cut cone C_n, then the inequality $v'.x \leqslant 0$ defines a facet of the even cut cone* EvC_m.

The bound $m \geqslant n+5$ is sharp since, for instance, the triangle inequality $x_{12} - x_{13} - x_{23} \leqslant 0$ is facet defining for C_3 and thus for EvC_n for $n \geqslant 8$, but not for EvC_6. Actually, this is the only facet of EvC_8 coming from some lower-dimensional cut cone.

4. Q-inequalities for the even cut cone

Given integers $b_1, \ldots, b_n$, $b = (b_1, \ldots, b_n)$, we consider the inequality:

$$Q(b).x = \sum_{1 \leqslant i < j \leqslant n} b_i b_j x_{ij} \leqslant 0 \tag{5}$$

We call it *Q-inequality*, or Q_σ-inequality, when we want to specify that the sum $b_1 + \cdots + b_n$ is equal to σ. The Q_0-inequalities, called negative-type inequalities in Functional Analysis, are valid for the cut cone C_n, but not facet inducing. The Q_1-inequalities are called *hypermetric* inequalities; they are valid for the cut cone C_n and contain large classes of facet inducing inequalities (see, e.g. [8–10, 15, 16]. From Theorem 3.1, for integers $b_1, \ldots, b_n$ with $\sum_{1 \leqslant i \leqslant n} b_i = 1$, if the inequality $Q(b_1, \ldots, b_n).x \leqslant 0$ defines a facet of C_n, then the inequality $Q(b_1, \ldots, b_n, 0, \ldots, 0).x \leqslant 0$ (with $p \geqslant 5$ additional zeros) defines a facet of EvC_{n+p}. On the other hand, we have the following negative result:

Proposition 4.1. *Let $b_1, \ldots, b_n$ be integers that are all odd except one and with $\sum_{1 \leqslant i \leqslant n} b_i = 1$. Then, the inequality* (5) *is valid, but not facet inducing for the even cut cone* EvC_n.

Proof. Assume for instance that b_n is even, and $b_1, \ldots, b_{n-1}$ are odd. Take an even cut $\delta(S)$ with, e.g. $n \notin S$. Then, $Q(b)(\delta(S)) = b(S)(1 - b(S)) = 0$ if and only if $b(S) = 0$. Therefore, every root of the inequality (5) is also a root of the Q_0-inequality $Q(b_1, \ldots, b_{n-1}, b_n - 1).x \leqslant 0$. Hence, the inequality (5) is not facet inducing. $\square$

For instance, $Q(1,\dots,1,-1,\dots,-1,0).x\leqslant 0$, with $k+1$ coefficients 1 and k coefficients -1, defines a facet of C_{2k+2}, but not of EvC_{2k+2}. Also, $Q(2,1,\dots,1,-1,\dots,-1).x\leqslant 0$, with k coefficients 1 and $k+1$ coefficients -1, defines a facet of C_{2k+2}, but not of EvC_{2k+2}.

In this section, we consider the class of Q_2-inequalities $Q(b).x\leqslant 0$ where all b_i are odd integers. They are, in a sense, an even analogue of the Q_1-inequalities (hypermetric inequalities) for the cut cone.

Proposition 4.2. *Given $b_1,\dots,b_n$ odd integers with $\sum_{1\leqslant i\leqslant n} b_i=2$, the inequality $Q(b).x\leqslant 0$ is valid for the even cut cone* EvC_n.

Proof. Take an even cut $\delta(S)$, i.e. $|S|$ is even. Then, $Q(b)(\delta(S))=b(S)(2-b(S))\leqslant 0$, since $b(S)\neq 1$, because $b(S)$ is an even integer. $\square$

Note that Proposition 4.2 remains trivially valid if one assumes that all b_i are even integers, but then the inequality $Q(b).x\leqslant 0$ is just a multiple of the (Q_1-)inequality $Q(b_1/2,\dots,b_n/2).x\leqslant 0$. If some of the b_i are even and some are odd, then validity is lost in general; for instance, $Q(2,2,1,1,-1,-1,-1,-1).x\leqslant 0$ is violated by the even cut $\delta(\{1,5\})$.

For instance, all facets of EvC_6 are in fact the 15 permutations of the facet induced by the Q_2-inequality $Q(1,1,1,1,-1,-1).x\leqslant 0$, while all facets of C_n, $n\leqslant 6$, arise from Q_1-inequalities. For $n=8$, EvC_8 has three (up to permutation) classes of facets coming from Q_2-inequalities, namely, $Q(1,1,1,1,1,-1,-1,-1).x\leqslant 0$, $Q(1,1,1,1,1,1,-1,-3).x\leqslant 0$ and $Q(3,1,1,1,-1,-1,-1,-1).x\leqslant 0$ (note that the last two are switching equivalent). Up to switching and permutation, EvC_8 has exactly 5 types of facets which do not arise from Q-inequalities; compare with the cut cone C_7 which also has 5 types of facets which do not come from Q-inequalities (for details, see [15]).

Theorem 4.3. *The inequality $Q(1,\dots,1,-1,\dots,-1).x\leqslant 0$, with $k+2$ coefficients 1 and k coefficients -1, defines a facet of* EvC_{2k+2}.

The proof is given in Section 6.2.

Remark 4.4. By switching the inequality $Q(1,\dots,1,-1,\dots,-1).x\leqslant 0$, with $k+2$ coefficients 1 and k coefficients -1, by the odd cut $\delta(\{1\})$, we deduce that the inequality $Q(1,\dots,1,-1,\dots,-1).x\leqslant -1$, with $k+1$ coefficients 1 and $k+1$ coefficients -1, defines a facet of the odd cut polytope OdP_{2k+2}. Hence, the Q_0-inequality $Q(1,\dots,1,-1,\dots,-1).x\leqslant 0$ (with the same number $k+1$ of coefficients 1 and -1) is not facet inducing for the cut polytope, neither for the even nor for the odd cut polytopes, but a suitable translation of this face is a facet of the odd cut polytope.

We conclude this section by mentioning generalizations of Q_2-inequalities in three directions: for even T-cuts, for σ-ary cuts and for even multicuts.

4.1. Q_2-inequalities for even T-cuts

For any n, let T be a subset of $[1,n]$ of even size. The following proposition contains Proposition 4.2 as the special case $T=[1,n]$.

Proposition 4.5. *Given integers $b_1,\ldots,b_n$ such that b_i is odd for $i\in T$, b_i is even for $i\notin T$ and $\sum_{1\leqslant i\leqslant n} b_i=2$, the inequality $Q(b_1,\ldots,b_n).x\leqslant 0$ is valid for the cone generated by the incidence vectors of the even T-cuts of the graph K_n.*

Proof. Take an even T-cut $\delta(S)$, i.e. $|S\cap T|$ is even, then $Q(b)(\delta(S))=b(S)(2-b(S))\leqslant 0$ since $b(S)=b(S\cap T)+b(S-T)\neq 1$, because both $b(S\cap T)$, $b(S-T)$ are even numbers. □

4.2. Q_2-inequalities for σ-ary cuts

Let $\sigma\geqslant 2$ be an integer and take $n\equiv 0(\operatorname{mod}\sigma)$. A cut $\delta(S)$ is called *σ-ary* if $|S|\equiv 0(\operatorname{mod}\sigma)$; so 2-ary cuts are even cuts. The following proposition contains Proposition 4.2 as the special case for $\sigma=2$.

Proposition 4.6. *Let $b_1,\ldots,b_n$ be integers such that $\sum_{1\leqslant i\leqslant n} b_i=\sigma$ and $b_i\equiv\beta(\operatorname{mod}\sigma)$ for $1\leqslant i\leqslant n$, where $\beta\in\{1,2,\ldots,\sigma-1\}$. Then, the inequality $Q(b).x\leqslant 0$ is valid for the cone generated by the incidence vectors of all σ-ary cuts of K_n.*

Proof. Take a σ-ary cut $\delta(S)$, i.e. $|S|\equiv 0(\operatorname{mod}\sigma)$. Then, $Q(b)(\delta(S))=b(S)(\sigma-b(S))\leqslant 0$ since $b(S)\neq 1,2,\ldots,\sigma-1$, because $b(S)\equiv|S|\beta\equiv 0(\operatorname{mod}\sigma)$. □

4.3. Q_2-inequalities for even multicuts

For n even, given a partition of $[1,n]$ into k subsets $S_1,\ldots,S_k$, the multicut $\delta(S_1,\ldots,S_k)$ is the set of edges of K_n whose endnodes belong to distinct classes S_i. We say that the multicut $\delta(S_1,\ldots,S_k)$ is even if all shores $S_1,\ldots,S_k$ are of even cardinality. The multicut (even multicut) polytope is the convex hull of the incidence vectors of all multicuts (all even multicuts). The Q_1-inequalities were extended to the multicut polytope as follows. Given $b_1,\ldots,b_n$ with $\sigma=\sum_{1\leqslant i\leqslant n} b_i\geqslant 1$, the inequality $Q(b)\leqslant\sigma(\sigma-1)/2$ is valid for the multicut polytope [14]. The Q_2-inequalities admit the following extension to the even multicut polytope (the link with Q_2-inequalities is transparent through the proof of Proposition 4.7).

Propositon 4.7. *Given odd integers $b_1,\ldots,b_n$ with $\sigma=\sum_{1\leqslant i\leqslant n} b_i\geqslant 2$, the inequality $Q(b)\cdot x\leqslant\sigma(\sigma-2)/2$ is valid for the even multicut polytope of the graph K_n.*

Proof. Take a multicut $\delta(S_1,\ldots,S_k)$; one easily checks that its incidence vector X_0 can be written as $(X_1+\cdots+X_n)/2$, where X_i is the incidence vector of the cut $\delta(S_i)$. Then

$Q(b)(\delta(S_1, \dots, S_k)) = (\sum_{1\leqslant i\leqslant k} Q(b)(\delta(S_i)))/2 = (\sum_{1\leqslant i\leqslant k} b(S_i)(\sigma - b(S_i)))/2 = \sigma/2(\sum_{1\leqslant i\leqslant k} b(S_i)) - (\sum_{1\leqslant i\leqslant k} b(S_i)^2)/2 = \sigma^2/2 - (\sum_{1\leqslant i\leqslant k} b(S_i)^2)/2$ which is less than or equal to $\sigma(\sigma-2)/2$ if and only if $\sum_{1\leqslant i\leqslant k} b(S_i)^2 \geqslant 2\sigma = \sum_{1\leqslant i\leqslant k} 2b(S_i)$. i.e. $\sum_{1\leqslant i\leqslant k} b(S_i)(2-b(S_i)) \leqslant 0$. The latter inequality indeed holds, since each member of the sum is nonpositive from Proposition 4.2. □

5. Isolator inequalities for the even cut cone

Each facet inducing inequality for the even cut cone which is not facet inducing for the cut cone is necessarly violated by some odd cuts. In this section, we consider such inequalities that are violated by exactly one cut; we call such inequalities *cut isolator* or simply *isolator* inequality. From the formula (1), it follows that, if an inequality $v.x \leqslant v_0$ is violated by the cut $\delta(S)$ only, then the inequality $v^S.x \leqslant v_0 - v.\delta(S)$ is violated by the zero cut only. In this section, we present some examples of isolator inequality yielding a facet of the even cut cone.

For any n, consider a partition of $[1,n]$ into the sets $\{1,n\}$, A and A', and define the inequality:

$$v_A.x = -(n-4)x_{1n} + \sum_{i\in A}(x_{1i} - x_{in}) + \sum_{i\in A'}(x_{in} - x_{1i}) \leqslant 0 \tag{6}$$

Lemma 5.1. *The inequality* (6) *is valid for all cuts except the cut* $\delta(A\cup\{n\})$. *Its roots are the cuts* $\delta(S)$ *for* S *subset of* $[2,n-1]$, $\delta(A\cup\{i,n\})$ *for* $i\in A'$ *and* $\delta(A-\{i\}\cup\{n\})$ *for* $i\in A$.

Proof. Take a cut $\delta(S)$ and w.l.o.g. assume that $1\in S$. If $n\in S$, then $v_A(\delta(S))=0$; hence, all cuts $\delta(T)$ with T subset of $[2,n-1]$ are indeed roots of $v_A.x\leqslant 0$. We now suppose that $n\notin S$. Set $a=|S\cap A|$ and $b=|S\cap A'|$. Then $v_A(\delta(S)) = -(n-4) - 2a + 2b + |A| - |A'| = 2(b-a+1-|A'|)$. The latter quantity is positive if and only if $a=0$ and $b=|A'|$, i.e. $\delta(S)=\delta(A+\{n\})$. The roots are obtained for $a=0$, $b=|A'|-1$, i.e. $\delta(S)=\delta(A\cup\{i,n\}))$ for $i\in A'$, or $a=1$, $b=|A'|$, i.e. $\delta(S)=\delta(A\cup\{n\}-\{i\})$ for $i\in A$. □

Theorem 5.2. *Given n even, $n\geqslant 6$, and a subset A of $[2,n-1]$ of even size, the inequality* (6) *defines a facet of the even cut cone* EvC_n.

Proof. Set $R_1 = \{\delta(\{i,j\})$ for $2\leqslant i\leqslant j\leqslant n-1$, $\delta(\{1,n\})$, $\delta(\{1,n,2,i\})$ for $3\leqslant i\leqslant n-1\}$ and $R_2 = \{\delta(A\cup\{i,n\})$ for $i\in A'$, $\delta(A-\{i\}\cup\{n\})$ for $i\in A\}$. So, $R_1\cup R_2$ is a set of $n(n-1)/2-1$ cuts defining roots of inequality (6). In order to prove that inequality (6) defines a facet of EvC_n, it suffices to prove that the family of the incidence vectors of the cuts in $R_1\cup R_2$ is linearly independent. For this, we take a linear combination of the incidence vectors of the cuts in $R_1\cup R_2$ which is equal to 0. viz.

$$y = \sum_{2\leqslant i<j\leqslant n-1} a_{ij}X^{\delta(\{i,j\})} + a_0 X^{\delta(\{1,n\})} + \sum_{3\leqslant i\leqslant n-1} b_i X^{\delta(\{1,n,2,i\})} + \sum_{i\in A'} c_i X^{\delta(A\cup\{i,n\})} + \sum_{i\in A} c_i X^{\delta(A-\{i\}\cup\{n\})} = 0.$$

By computing the value of component y_{1n}, we deduce that:

(a) $\sum_{2\leqslant i\leqslant n-1} c_i=0$

By computing $y_{1i}-y_{in}$ for $i\in A$ or $i\in A'$, we deduce that $c_i=0$ and also the following relations:

(b) $a_0+\sum_{2\leqslant k\leqslant n-1, k\neq i} a_{ik}+\sum_{3\leqslant i\leqslant n-1} b_i-b_i=0$ for $i\in[3,n-1]$,

(c) $a_0+\sum_{3\leqslant k\leqslant n-1} a_{2k}=0$.

By summing (c) and (b) over $3\leqslant i\leqslant n-1$, we deduce that:

(d) $(n-2)a_0+2\sum_{2\leqslant i<j\leqslant n-1} a_{ij}+(n-4)(\sum_{3\leqslant i\leqslant n-1} b_i)=0$.

By computing the scalar product of the all-ones vector with y, we obtain:

(e) $2(n-2)\,(a_0+\sum_{2\leqslant i<j\leqslant n-1} a_{ij})+4(n-4)(\sum_{3\leqslant i\leqslant n-1} b_i)=0$.

By computing y_{ij} for $3\leqslant i\leqslant j\leqslant n-1$ we obtain:

(f) $\sum_{2\leqslant k\leqslant n-1, k\neq i} a_{ik}+\sum_{2\leqslant k\leqslant n-1, k\neq j} a_{jk}-2a_{ij}+b_i+b_j=0$ for $3\leqslant i<j\leqslant n-1$;

also, we have:

(g) $\sum_{2\leqslant k\leqslant n-1, k\neq 2} a_{2k}+\sum_{2\leqslant k\leqslant n-1, k\neq i} a_{ik}-2a_{2i}+\sum_{3\leqslant i\leqslant n-1} b_i-b_i=0$ for $3\leqslant i\leqslant n-1$.

Using (b) and (c), we deduce from (f) and (g) that:

(h) $b_i+b_j-a_{ij}-\sum_{3\leqslant i\leqslant n-1} b_i-a_0=0$ for $3\leqslant i<j\leqslant n-1$,

(i) $a_{2i}+a_0=0$ for $3\leqslant i\leqslant n-1$.

From (c) and (i), we deduce that $a_0=0$ and $a_{2i}=0$ for $3\leqslant i\leqslant n-1$. Relations (d) and (e) imply that $\beta:=\sum_{3\leqslant i\leqslant n-1} b_i=-\sum_{2\leqslant i\leqslant j\leqslant n-1} a_{ij}$. From (h), $a_{ij}=b_i+b_j-\beta$ for $3\leqslant i<j\leqslant n-1$, implying that $\sum_{2\leqslant j\leqslant n-1, j\neq i} a_{ij}=(n-4)(b_i-\beta)$. From (g), $\sum_{2\leqslant j\leqslant n-1, j\neq i}=a_{ij}=b_i-\beta$. Therefore, $b_i=\beta$ for $3\leqslant i\leqslant n-1$. We deduce that $b_i=0$ for $3\leqslant i\leqslant n-1$ and thus $a_{ij}=0$ for $3\leqslant i<j\leqslant n-1$. □

If we switch inequality (6) by a cut $\delta(S)$ with S subset of $[2,n-1]$, we obtain an inequality of the same form. If we switch (6) by a cut $\delta(S)$ with $|S\cap\{1,n\}|=1$, we obtain an inequality of the form

$$w_B.x=(n-4)x_{1n}+\sum_{i\in B}(x_{1i}+x_{in})-\sum_{i\in B'}(x_{1i}+x_{in})\leqslant 2(|B|-1), \tag{7}$$

where B is a subset of $[2,n-1]$ and $B'=[2,n-1]$-B (namely, $B=(A\cap S)\cup(A'-S)$ if $1\in S$).

The inequality (7) is violated only by the cut $\delta(B)$. Therefore, we deduce from Theorem 5.2 that, for any subset B of $[2,n-1]$ of odd size, the inequality (7) defines a facet of EvP_n. Consequently, the inequalities (6), for any subset A of odd size of $[2,n-1]$, and (7), for any subset B of $[2,n-1]$ of even size, induce facets of the odd cut polytope OdP_n. Note that, for $n=8$, up to permutation, there are exactly five facets of EvP_8 of the type (6) or (7), namely, $v_A.x\leqslant 0$ for A subset of $[2,7]$ of size 0, 2, 4 and $w_B.x\leqslant 2(|B|-1)$ for B subset of $[2,7]$ of size 1 or 3.

We conclude this section by mentioning some other classes of isolator inequalities; actually, for $n=6$, these examples together with (6) are (up to switching) the only isolator inequalities that define facets of the polytope whose vertices are the incidence vectors of all nonzero cuts of K_6.

Example 5.3. Let C denote a cycle on n nodes. Then, the inequality $\sum_{ij\in C} x_{ij}\geqslant 2$ is valid for all cuts except the zero cut. Actually, it defines a facet of the polytope whose vertices are the incidence vectors of all nonzero cuts [13]. So if C is the cycle $(1,2,\ldots,n)$, for any odd s, $1\leqslant s\leqslant n$, the inequality $2(x_{n-1n}+x_{s+1\,s+2})-\sum_{ij\in C}x_{ij}\leqslant 0$ is valid for the even cut cone.

Example 5.4. Let $b_1,\ldots,b_p,b_{p+1}>0$, $b_{p+2},\ldots,b_n<0$ be integers with $\sum_{1\leqslant i\leqslant n}b_i=3$, and let C denote the cycle $(1,\ldots,p)$. The inequality $Q(b_1,\ldots,b_n).\ x-\sum_{ij\in C}x_{ij}\leqslant 0$ is valid for all cuts except the cut $\delta(\{p+1\})$. Indeed, if $\delta(S)$ violates the above inequality, then $b(S)(3-b(S))=2$ and $|\delta(S)\cap C|=0$, i.e. $b(S)=1$ or 2, implying that $S=\{p+1\}$. So, for n even, it is valid for the even cut cone.

Example 5.5. The inequality

$$\sum_{1\leqslant i\leqslant p-1}(2x_{2i-1,2i}-x_{2i+1,2i-1}-x_{2i+1,2i}-x_{2i+2,2i-1}-x_{2i+2,2i})$$
$$-(x_{2p-1,2p}-x_{2p-1,2p+1}-x_{2p,2p+1})-(x_{12}+x_{1,2p+2}+x_{2,2p+2})\leqslant -2$$

is valid for all cuts except the zero cut.

Example 5.6. The inequality $x_{12}+x_{13}+x_{23}-(x_{24}+x_{34}+x_{36}+x_{56}+x_{26})\leqslant 0$ is valid for all cuts of K_6 except the cut $\delta(\{1\})$.

6. Proofs

We establish some lemmas that will be used in the proofs. We shall use the following notation. For disjoint subsets A,B of $[1,n]$, $\delta(A,B)$ denotes the set of edges with one endnode in A and the other endnode in B; So $\delta(S)=\delta(S,[1,n]-S)$. If b is a vector of $\mathbf{R}^{n(n-1)/2}$, we shall sometimes, for short, denote $b(\delta(A,B))=\sum_{i\in A,\,j\in B}b_{ij}$ by $b(A,B)$. For $i\notin S$, we set $S+i=S\cup\{i\}$.

Lemma 6.1. *Let* $b\in\mathbf{R}^{n(n-1)/2}$, $b_0\in\mathbf{R}$, $i,j,k\in[1,n]$ *and let* S *be a subset of* $[1,n]-\{i,j,k\}$ *such that* $b(\delta(S\cup\{i\}))=b(\delta(S\cup\{j\}))=b(\delta(S\cup\{k\}))=b(\delta(S\cup\{i,j,k\})=b_0$. *Then,* $b_{ij}+b_{ik}+b_{jk}=b_0-b(\delta(S))$.

Proof. Set $A=[1,n]-S\cup\{i,j,k\}$. Then, $2b_0=b(\delta(S+i))+b(\delta(S+j))+b(\delta(S+k))-b(\delta(S\cup\{i,j,k\}))=2b(A,S)+2b(S,\{i,j,k\})+2(b_{ij}+b_{ik}+b_{jk})$, implying that $b_0=b(\delta(S))+b_{ij}+b_{ik}+b_{jk}$. □

Lemma 6.2. *Let* $b\in\mathbf{R}^{n(n-1)/2}$, $b_0\in\mathbf{R}$, $i,j,k,l\in[1,n]$ *and let* S *be a subset of* $[1,n]-\{i,j,k,l\}$ *such that* $b(\delta(S\cup\{i,j\}))=b(\delta(S\cup\{i,k\}))=b(\delta(S\cup\{j,l\}))=b(\delta(S\cup\{k,l\}))=b(\delta(S\cup\{i,l\}))=b(\delta(S\cup\{j,k\}))=b_0$ *and let one of the following two assertions hold*:

(i) $b(\delta(S))=b_0$,

(ii) $b(\delta(S\cup\{i,j,k,l\}))=b_0$.

Then, $b_{ij}+b_{kl}=b_{ik}+b_{jl}=b_{il}+b_{jk}$. *Moreover, if both* (i) *and* (ii) *hold, then* $b_{ij}+b_{kl}=0$.

Proof. Set $A=[1,n]-S\cup\{i,j,k,l\}$. Under assumption (i), we have:

(a) $0=b(\delta(S\cup\{i,j\}))-b(\delta(S))=b(A,\{i,j\})-b(S,\{i,j\})+b(\{i,j\},\{k,l\})$.

Similarly,

(b) $0=b(A,\{i,k\})-b(S,\{i,k\})+b(\{i,k\},\{j,l\})$,

(c) $0=b(A,\{j,l\})-b(S,\{j,l\})+b(\{j,l\},\{i,k\})$,

(d) $0=b(A,\{k,l\})-b(S,\{k,l\})+b(\{k,l\},\{i,j\})$.

By computing (a)$-$(b)$-$(c)$+$(d), we obtain that $0=b_{ik}+b_{jl}-b_{ij}-b_{kl}$. The result follows by symmetry. The case when (ii) is assumed is treated similarly; for instance, (a) is replaced by:

(a′) $0=b(\delta(S\cup\{i,j,k,l\}))-b(\delta(S\cup\{i,j\}))=b(A,\{k,l\})-b(S,\{k,l\}))-b(\{i,j\},\{k,l\}))$.

If both (i) and (ii) hold, then, relations (a) and (a′) yield that $b(\{i,j\},\{k,l\})=0$, implying $b_{ik}+b_{jl}=0$. □

Lemma 6.3 [3]. *Let $b\in\mathbf{R}^{n(n-1)/2}$, S,I,J,K be disjoint subsets of $[1,n]$ such that $b(\delta(S\cup J))=b(\delta(S\cup K))=b(\delta(S\cup I\cup J))=b(\delta(S\cup I\cup K))$, then $b(I,J)=b(I,K)$ holds.*

6.1 Proof of Theorem 3.1

Set $m=n+p$, $p\geqslant 5$. Set $N=[1,n]$, $P=[1',p']$ and $M=N\cup P$. So the cut cone C_n is defined on the n nodes of N, and the even cut polytope EvP_m is defined on the m nodes of M. The inequality $v'.x\leqslant 0$ is trivially valid for EvP_m. In order to prove that it defines a facet of $\mathrm{EvP}_{m'}$ take an inequality $b.x\leqslant 0$ that is valid for EvP_m and satisfies $\{x\in\mathrm{EvP}_m\colon v'.x=0\}\subseteq\{x\in\mathrm{EvP}_m\colon b.x=0\}$; we show that $b=\alpha v'$ for some scalar α. We distinguish two cases: n,p odd and n,p even.

Let us first assume that n and p are odd.

Claim 6.4. $b_{ij}=0$ *for* $i,j\in P$.

Proof. For $S=\emptyset$, the cuts $\delta(S)$, $\delta(S\cup\{i,j\})$, $\delta(S\cup\{i,j,k,l\})$ are even cuts of C_m and define roots of $b.x.\leqslant 0$, for all distinct points i,j,k,l of P. We deduce from Lemma 6.2 that $b_{ij}+b_{kl}=0$ holds. Since $p\geqslant 5$, take another point h of P; then we deduce that $b_{ij}+b_{kl}=b_{ij}+b_{kh}=0$; also, $b_{kl}+b_{ih}=b_{jl}+b_{ih}=b_{jl}+b_{kh}=0$, implying that $b_{kh}=0$. □

Take any root $\delta(S)$ of $v.x\leqslant 0$ in C_n, e.g. $|S|$ is odd while $|S'|$ is even, setting $S'=[1,n]-S$; so

$$b(\delta(S'))=0 \tag{8}$$

On the other hand, $\delta(S\cup A)$ is an even cut and is root of $v'.x\leqslant 0$ for any odd subset A of P; so we deduce from Lemma 6.1 that

$$b(\delta(S))=0 \tag{9}$$

Relations (8) and (9) imply

$$b(S,P)=b(S',P)=b(N,P)/2 \tag{10}$$

Now, applying (10) with $S=N$, we deduce that $b(N,P)=0$ and thus

$$b(S,P)=0 \tag{11}$$

Take $i\in P$; then, using the above relations and $0=b(\delta(S+i))$, we deduce that $b(i,S)=b(i,N)/2$. Hence, for $S=N$, we deduce that

$$b(i,S)=0 \tag{12}$$

Since $v.x\leqslant 0$ defines a facet of C_n, we can find $n-1$ subsets S_i of $[2,n]$ such that $\delta(S_i)$ is root of $v.x\leqslant 0$ and the incidence vectors of the sets S_i are linearly independent. Since relation (12) holds for any subset S for which $\delta(S)$ is root of $v.x\leqslant 0$, we deduce that $b_{i2}=\cdots=b_{in}=0$ which, together with $b(i,N)=0$, implies that $b_{i1}=0$. Therefore, by symmetry, we deduce that $b_{ij}=0$ for $i\in N$, $j\in P$.

Let c denote the projection of b on $\mathbf{R}^{n(n-1)/2}$, so $b=(c,0,\ldots,0)$. The inequality $c.x\leqslant 0$ is valid for C_n and admits all roots of $v.x\leqslant 0$ as roots, implying that $c=\alpha v$. This concludes the proof in the case when n and p are odd.

We now consider the case when n and p are even. Claim 6.4 remains valid. Take first a root $\delta(S)$ of $v.x\leqslant 0$ in C_n with $|S|$ and $|S'|$ even. Then, $b(\delta(S))=b(\delta(S'))=0$, implying that $b(S,P)=b(N,P)/2$; so, for $S=N$, we obtain that $b(S,P)=b(N,P)=0$. Given i,j in P, then $b(\delta(S\cup\{i,j\}))=b(\delta(S'\cup\{i,j\}))=0$, implying that $b(S,\{i,j\})=b(N,\{i,j\}))/2$ and thus $b\{S,\{i,j\})=b(N\{i,j\})=0$ from which one obtains

$$b(S,i)=b(N,i)=0 \tag{13}$$

for any even subset S of $(1,n)$ defining a root of $v.x\leqslant 0$.

Take now a root $\delta(S)$ of $v.x\leqslant 0$ with $|S|$ and $|S'|$ odd. Using Lemma 6.1, we easily deduce that $b(\delta(S))=b(\delta(S'))=0$. Since $b(N,P)=0$, we obtain that $b(S,P)=0$. As before, we obtain that $b(i,S)=0$ for any $i\in P$. Therefore, relation (13) holds for all sets S defining a root of $v.x\leqslant 0$. One finishes the proof in the same way as before. □

6.2 *Proof of Theorem 4.4*

In order to show that the inequality $Q(1,\ldots,1,-1,\ldots,-1).x\leqslant 0$ defines a facet of EvP_n, we take an inequality $b.x\leqslant 0$ valid for EvP_n such that $\{x\in\mathrm{EvP}_n$: $Q(1,\ldots,1,-1,\ldots,-1).x=0\}\subseteq\{x\in\mathrm{EvP}_n$: $b.x=0\}$, i.e. $b(\delta(S))=0$ whenever $b(S)=0$ or 2, and we prove that $b.x$ is a multiple of $Q(1,\ldots,1,-1,\ldots,-1).x$. Set $Q(1,\ldots,1,-1,\ldots,-1)=Q(b)$ and denote by P the points i with $b_i=1$ and by Q the points i with $b_i=-1$, $P=[1,k+2]$, $Q=[1',k']$. We can assume that $k\geqslant 4$, since the cases $k=2,3$ were checked directly using computer.

Claim 6.5. *$b_{ij}=\beta$ for all i,j in P.*

Proof. We can apply Lemma 6.1 with $S=\{i'\}$ for $i'\in Q$, $i,j,k\in P$ and obtain that $b_{ij}+b_{ik}+b_{jk}=-b(\delta(\{i'\})$ depends only on the choice of i'. Therefore, if h is another point of P, then $b_{ij}+b_{ik}+b_{jk}=b_{ij}+b_{ih}+b_{jh}$, implying

$$b_{ik}+b_{jk}=b_{ih}+b_{jh}. \tag{14}$$

We now apply Lemma 6.2 (under assumption (ii)) to $S=\{1',2'\}$, i,j,k,h in P and obtain

$$b_{ih}+b_{jk}=b_{ik}+b_{jh} \tag{15}$$

Relations (14) and (15) imply that $b_{jk}=b_{jh}$ and thus $b_{ij}=\beta$ for all i,j in P.

Claim 6.6. *$b_{ij}=\alpha$ for all i,j in Q.*

The proof is similar to that of Claim 6.5.

Then, take points i,j,k,h in P and i',j' in Q. We can apply Lemma 6.3 to $S=\emptyset$, $I=\{i',j'\}$, $J=\{i,j\}$ and $K=\{k,h\}$; we deduce that $b(\{i',j'\},\{i,j\})=b(\{i',j'\},\{k,h\})$. Let l be another point of P; similarly, we have that $b(\{i',j'\},\{i,j\})=b(\{i',j'\},\{k,l\})$, implying that $b(\{i',j'\},\{h\})=b(\{i',j'\},\{l\})$ is, therefore, equal to some constant number $\beta_{i',j'}$. We saw above (in the proof of Claim 6.5) that $b(\delta(\{i'\}))=-3\beta$; therefore,

$$\begin{aligned} &b(\delta(\{i'\}))+b(\delta(\{j'\}))\\ &\quad=-6\beta=2(k-1)\alpha+\textstyle\sum_{i\in P}b_{ii'}+b_{ij'}\\ &\quad=2(k-1)\alpha+(k+2)\beta_{i'j'}, \end{aligned}$$

from which we deduce that $\beta_{i'j'}=\beta'$ does not depend on i',j'. Now, $b(\delta(\{1,2\}))=0$, yielding that $\beta'=-2\beta$ and the preceeding relation yields that $\alpha=\beta$. Moreover, $b_{ii'}=-\beta$ for $i\in P$, $i'\in Q$. $\square$

Acknowledgements

We are very grateful to Komei Fukuda who kindly provided us the full description obtained by computer of the even cut polytope on 8 nodes, as well as several additional computations.

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